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Quasi-Fibonacci oscillators

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Abstract

We study the properties of the sequences of the energy eigenvalues for some generalizations of q -deformed oscillators including the p, q -oscillator, and the three-, four- and five-parameter deformed oscillators given in the literature. It is shown that most of the considered models belong to the class of so-called Fibonacci oscillators for which any three consecutive energy levels satisfy the relation $E_{n+1} = \lambda E_n + \rho E_{n-1}$ with real constants λ, ρ . On the other hand, for a certain μ -oscillator known since 1993, we prove its non-Fibonacci nature. Possible generalizations of the three-term Fibonacci relation are discussed, among which for the μ -oscillator we choose, as the most adequate, the so-called quasi-Fibonacci (or local Fibonacci) property of the energy levels. The property is encoded in the three-term quasi-Fibonacci (QF) relation with the non-constant, n -dependent coefficients λ and ρ . Various aspects of the QF relation are elaborated for the μ -oscillator and some of its extensions.

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1. Introduction

In 1991, the two-parameter family of p, q -deformed quantum oscillators was introduced [1]. Soon after, in [2] the family was named ‘Fibonacci oscillators’, due to the basic property of the family encoded in the three-term linear recurrence relation [2]

$$E_{n+1} = \lambda E_n + \rho E_{n-1}, \quad n \geq 1, \quad \lambda, \rho \in \mathbb{R}, \quad (1)$$

for any three consecutive values of energy levels from the spectrum of respective deformed quantum oscillators, with the definite coefficients λ and ρ that depend on the specific model of a deformed oscillator. The usual Fibonacci numbers form the sequence generated by the equation like (1) in which $\lambda = \rho = 1$ and the first two members of the sequence are fixed as 1 and 1; the property states that each number in the sequence is the sum of the two preceding ones.

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Following [2], in our paper we call the Fibonacci oscillators those oscillators whose any three energies E_{n-1} , E_n , E_{n+1} satisfy the Fibonacci relation (FR) given by (1). They form the Fibonacci class of oscillators. There exist a number of known oscillators that are the Fibonacci class of oscillators. For instance, the usual harmonic oscillator certainly is the Fibonacci oscillator in the sense of equation (1) with $\lambda = 2$ and $\rho = -1$. As already mentioned, the family of p , q -oscillators, which contains well-known models of one-parameter q -oscillators, belongs to the Fibonacci class. However, the question naturally arises about the possible existence of other models of deformed oscillators which satisfy (1), i.e. possess the Fibonacci property (FP). Recently, in conjunction with generalized Heisenberg algebras [3], possible extensions of the FP were studied, either in the direction of nonlinearization of the relation, or toward the so-called k -step extension (in particular, Tribonacci sequence), see [4]. Also, in [5] some classes of non-Fibonacci (namely so-called k -bonacci) oscillators have been explored.

Our goal in this paper is two-fold. First, we study some multi-parameter extended families of deformed oscillators from the viewpoint of possessing the FP, including three- and four-parametric ones from [6–8] as well as their five-parameter extended model given in [9]. We prove that all these multi-parameter deformations belong to the class of Fibonacci oscillators. On the other hand, we examine a certain (as yet not well studied) deformation called the μ -oscillator, which appeared in [10], and present the proof that this model lives outside the Fibonacci class (that is, it is non-Fibonacci one). This conclusion served as motivation for us to explore the possible (still linear) extensions² of the FR and to find the most adequate and natural form of generalization among them which is suitable for the μ -oscillator. In other words, our second goal is to provide a proper generalization of the FP that is adequate for both the μ -oscillator and its several multi-parameter extensions. The employed particular generalization of the FP can be termed the ‘quasi-Fibonacci (QF)’ property. Correspondingly, the μ -oscillator [10] and its appropriate extensions belong to the class of QF oscillators. In some sense, this can also be viewed as the deformation of the quantum oscillator whose energy values form a ‘locally Fibonacci’ sequence of values.

2. Deformed oscillators of the Fibonacci class

In this section, we examine the energy spectra of the models of multi-parameter deformed oscillators from [6–9], from the viewpoint of possessing the FP. But it will be useful to first recall the main facts about some well-known models of deformed oscillators. A general approach for the treatment of deformed oscillators involves the notion [12–14] of the structure function $f(N)$ or $\varphi(N)$ that serves to define the deformation. In terms of the structure function $\varphi(N)$ given by the equalities

$$\varphi(N) = a^\dagger a, \quad \varphi(N+1) = a a^\dagger, \quad (2)$$

the most general commutation relation can be written in the form

$$\varphi(N+1) - F(N)\varphi(N) = G(N),$$

with $F(N)$ and $G(N)$ the real functions. Every special choice of $F(N)$ and $G(N)$ provides, through solving the latter equation [13], the particular explicit form of the structure function $\varphi(N)$ and thus the corresponding model of the deformed oscillator. With fixed $\varphi(N)$, the commutation relations for the generating elements a , a^\dagger and N of the concrete model of the deformed oscillator are given as

$$[N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \quad (3)$$

² For comparison see e.g. [11] for an interesting example of a nonlinear three-term relation giving the values of the (non-equidistant) area spectrum of a 5D Gauss–Bonnet black hole.

$$aa^\dagger - a^\dagger a = \varphi(N + 1) - \varphi(N). \tag{4}$$

That means that we use the form of a basic commutation relation such that $F(N) = 1$ and $G(N) = \varphi(N + 1) - \varphi(N)$. In this case, fixation of the structure function $\varphi(N)$ completely specifies the (deformed oscillator) model.

2.1. Well-known Fibonacci oscillators

Let us recall some of the most popular models of deformed oscillators which involve one or two deformation parameters and belong to the Fibonacci class of oscillators.

- *Arik–Coon (AC) model* [15]. This most early known model is given by the commutation relations (3) and

$$aa^\dagger - qa^\dagger a = 1. \tag{5}$$

In what follows, for all the models we use the same definition of the Hamiltonian as for the usual harmonic oscillator:

$$H = \frac{1}{2}(aa^\dagger + a^\dagger a), \tag{6}$$

where $\hbar\omega = 1$ is meant. Using the appropriately modified (deformed) version of the Fock space [13, 14] wherein $H|n\rangle = E_n|n\rangle$ and $\varphi_{AC}(N)|n\rangle = \varphi_{AC}(n)|n\rangle$, the energy spectrum of the AC model is

$$E_n = \frac{1}{2}(\varphi_{AC}(n + 1) + \varphi_{AC}(n)) = \frac{1}{2}([n + 1]_{AC} + [n]_{AC}) = \frac{1}{2} \left(\frac{q^{n+1} - 1}{q - 1} + \frac{q^n - 1}{q - 1} \right), \tag{7}$$

where the definition of the AC-type q -bracket (for X either a number or an operator) and the structure function is given as

$$[X]_{AC} = \frac{q^X - 1}{q - 1} \Leftrightarrow \varphi_{AC}(n) = \frac{q^n - 1}{q - 1} = [n]_{AC} \tag{8}$$

so that the latter satisfies (5).

One can easily check that the AC-oscillator energies (7) satisfy the FR (1) if $\lambda = 1 + q$ and $\rho = -q$.

- *Biedenharn–Macfarlane (BM) model* [16, 17]. This model is determined by the commutation relations (3) and the relations

$$aa^\dagger - qa^\dagger a = q^{-N}, \quad aa^\dagger - q^{-1}a^\dagger a = q^N. \tag{9}$$

The corresponding structure function which solves (9) is

$$\varphi_{BM}(N) = \frac{q^N - q^{-N}}{q - q^{-1}} \equiv [N]_{BM}, \tag{10}$$

and the energy spectrum reads

$$E_n = \frac{1}{2}(\varphi_{BM}(n + 1) + \varphi_{BM}(n)) = \frac{1}{2} \left(\frac{q^{n+1} - q^{-(n+1)}}{q - q^{-1}} + \frac{q^n - q^{-n}}{q - q^{-1}} \right). \tag{11}$$

The BM oscillator satisfies the FR (1) with $\lambda = q + q^{-1}$ and $\rho = -1$.

- *p, q -deformed oscillator* [1]. This model is determined by the commutation relations (3) and the relations

$$aa^\dagger - qa^\dagger a = p^N, \quad aa^\dagger - pa^\dagger a = q^N, \tag{12}$$

possessing the $q \leftrightarrow p$ symmetry. The relevant structure function satisfying (12) is

$$\varphi_{p,q}(N) = \frac{q^N - p^N}{q - p} \equiv [N]_{p,q}, \tag{13}$$

and hence the energy spectrum reads

$$E_n = \frac{1}{2} \left(\frac{q^{n+1} - p^{n+1}}{q - p} + \frac{q^n - p^n}{q - p} \right). \tag{14}$$

The model of the p, q -deformed oscillator is popular enough: it finds a number of interesting applications, see e.g. [18–25]. From it, by setting $p = 1$, the AC-model is retrieved. At $p = q^{-1}$, we recover the BM-type oscillator. Finally, the peculiar case of $p = q$ corresponds to the so-called Tamm–Dancoff (TD) deformed oscillator [26, 27]. This model, though not so popular as the former two, nevertheless possesses some interesting properties. More details concerning the TD-type oscillator model, including some of its properties, can be found in [28]. Note that the TD oscillator also arises as a particular one-dimensional case in the $SU(d)$ covariant d -dimensional so-called q -deformed Newton oscillator [29].

With (14), the two-parameter p, q -model of deformed oscillators satisfies the FR (1) for $\lambda = q + p$ and $\rho = -qp$. Accordingly, the AC-oscillator is the Fibonacci oscillator with $\lambda = 1 + q$ and $\rho = -q$, whereas the energy levels of the BM-oscillator satisfy (1) if we take $\lambda = q + q^{-1}$ and $\rho = -1$. Finally, to confirm that the TD-model of the deformed oscillator does satisfy (1), we put $\lambda = 2q$ and $\rho = -q^2$.

2.2. A family of multi-parameter deformed oscillators

In this subsection, we focus on those deformed multi-parameter oscillators which possess more than two deformation parameters in their defining relations and thus generalize the q - and p, q -deformed oscillators. It is of special interest to examine the properties of such deformed oscillators in dependence on the number of parameters. Our main goal is to examine whether these models of oscillators satisfy the FR (1), and in case they do not, to search for a way of appropriately modifying the FR. As an example of multi-parameter oscillators, consider the (p, q, α, β, l) -deformed oscillator family formulated in [9] and given by the defining relations

$$aa^\dagger - q^l a^\dagger a = p^{-\alpha N - \beta}, \quad aa^\dagger - p^{-l} a^\dagger a = q^{\alpha N + \beta}.$$

Here p, q, α, β, l are the parameters of deformation. The creation and annihilation operators act upon the Fock space basis state $|n\rangle$ by the formulae

$$aa^\dagger |n\rangle = \frac{q^{\alpha n + \beta + l} - p^{-(\alpha n + \beta + l)}}{q^l - p^{-l}} |n\rangle, \quad a^\dagger a |n\rangle = \frac{q^{\alpha n + \beta} - p^{-(\alpha n + \beta)}}{q^l - p^{-l}} |n\rangle. \tag{15}$$

With the Hamiltonian as in (6), the eigenvalues which form the energy spectrum are

$$E_n = \frac{1}{2} \left(\frac{q^{\alpha n + \beta + l} - p^{-(\alpha n + \beta + l)}}{q^l - p^{-l}} + \frac{q^{\alpha n + \beta} - p^{-(\alpha n + \beta)}}{q^l - p^{-l}} \right). \tag{16}$$

Unlike the models mentioned above, for this five-parameter model, the formula for E_n cannot be written as $E_n = \frac{1}{2}(\varphi_{n+1} + \varphi_n)$, but rather as $E_n = \frac{1}{2}(\varphi_n + \varphi_{n+\frac{l}{\alpha}})$; the presentation $E_n = \frac{1}{2}(\varphi_n + \varphi_{n+1})$ is retrieved only in the four-parameter case wherein $l = \alpha$.

The models of deformed oscillators examined in subsection 2.1 obviously follow from the (p, q, α, β, l) -deformed oscillator as particular cases. Moreover, the five-parameter family reduces at $l = 1$ to the four-parameter deformed oscillator treated in [7, 8] which in turn goes over into the three-parameter deformed quantum oscillators first formulated in [6].

2.3. Fibonacci property of multi-parameter oscillator models

The constants λ and ρ in (1) for the AC, BM and p, q -models of deformed oscillators given in subsection 2.1 can be deduced easily by solving the corresponding systems of equations. The principal feature of those models is that the coefficients λ, ρ are indeed constant, i.e. they do not depend on the eigenvalue n of the (particle number) operator N . Suppose that (1) is true for the extended three-, four- and five-parameter oscillators. Then the following statement is true.

Proposition 1. *The five-parameter deformed oscillator whose energies $E_n, n \geq 0$, are given by (16) satisfies, with certain constants λ and ρ , each of the following two relations:*

$$\begin{cases} E_{n+1} = \lambda E_n + \rho E_{n-1}, \\ E_{n+2} = \lambda E_{n+1} + \rho E_n, \end{cases} \quad (17)$$

and thus belongs to the class of Fibonacci oscillators.

The proof proceeds through solving for λ and ρ the system (17) taking into account (16). Recall that these λ and ρ should not depend on n . As follows from (17),

$$\lambda = \frac{E_{n+2}E_{n-1} - E_{n+1}E_n}{E_{n+1}E_{n-1} - E_n^2}, \quad \rho = \frac{E_{n+1}^2 - E_{n+2}E_n}{E_{n+1}E_{n-1} - E_n^2}. \quad (18)$$

Inserting from (16) the expression for $E_j, j = n - 1, n, n + 1, n + 2$, we obtain $\lambda = B/C$ where $B = B(n; p, q, \alpha, \beta, l), C = C(n; p, q, \alpha, \beta, l)$ and

$$\begin{aligned} B &= \left(\frac{q^{\alpha(n+2)+\beta+l} - p^{-\alpha(n+2)-\beta-l} + q^{\alpha(n+2)+\beta} - p^{-\alpha(n+2)-\beta}}{q^l - p^{-l}} \right) \\ &\quad \times \left(\frac{q^{\alpha(n-1)+\beta+l} - p^{-\alpha(n-1)-\beta-l} + q^{\alpha(n-1)+\beta} - p^{-\alpha(n-1)-\beta}}{q^l - p^{-l}} \right) \\ &\quad - \left(\frac{q^{\alpha(n+1)+\beta+l} - p^{-\alpha(n+1)-\beta-l} + q^{\alpha(n+1)+\beta} - p^{-\alpha(n+1)-\beta}}{q^l - p^{-l}} \right) \\ &\quad \times \left(\frac{q^{\alpha n+\beta+l} - p^{-\alpha n-\beta-l} + q^{\alpha n+\beta} - p^{-\alpha n-\beta}}{q^l - p^{-l}} \right), \\ C &= \left(\frac{q^{\alpha(n+1)+\beta+l} - p^{-\alpha(n+1)-\beta-l} + q^{\alpha(n+1)+\beta} - p^{-\alpha(n+1)-\beta}}{q^l - p^{-l}} \right) \\ &\quad \times \left(\frac{q^{\alpha(n-1)+\beta+l} - p^{-\alpha(n-1)-\beta-l} + q^{\alpha(n-1)+\beta} - p^{-\alpha(n-1)-\beta}}{q^l - p^{-l}} \right) \\ &\quad - \left(\frac{q^{\alpha n+\beta+l} - p^{-\alpha n-\beta-l} + q^{\alpha n+\beta} - p^{-\alpha n-\beta}}{q^l - p^{-l}} \right)^2. \end{aligned}$$

Calculation shows remarkable cancelation of the n -dependence; we get $B/C = B'/C'$ where

$$\begin{aligned} B' &= -q^{3\alpha} p^{2\alpha+l} - q^{3\alpha} p^{2\alpha} - q^{3\alpha+l} p^{2\alpha+l} - q^{3\alpha+l} p^{2\alpha} - p^{-\alpha} - p^{-\alpha+l} - q^l p^{-\alpha} - q^l p^{-\alpha+l} \\ &\quad + q^\alpha p^l + q^\alpha + q^{\alpha+l} p^l + q^{\alpha+l} + q^{2\alpha} p^{\alpha+l} + q^{2\alpha} p^\alpha + q^{2\alpha+l} p^{\alpha+l} + q^{2\alpha+l} p^\alpha, \\ C' &= -p^l - 1 - q^l p^l - q^l + 2q^\alpha p^{\alpha+l} + 2q^\alpha p^\alpha + 2q^{\alpha+l} p^{\alpha+l} \\ &\quad + 2q^{\alpha+l} p^\alpha - q^{2\alpha} p^{2\alpha+l} - q^{2\alpha} p^{2\alpha} - q^{2\alpha+l} p^{2\alpha+l} - q^{2\alpha+l} p^{2\alpha}. \end{aligned}$$

Moreover, it is easy to check that $(q^\alpha + p^{-\alpha})C' = B'$. Hence, finally

$$\lambda = q^\alpha + p^{-\alpha}. \quad (19)$$

The same procedure applied to ρ yields $\rho = D/E$ where

$$D = q^{3\alpha} p^\alpha + q^{3\alpha+l} p^\alpha + q^\alpha p^{-\alpha} + q^\alpha p^{-\alpha+l} + q^{\alpha+l} p^{-\alpha} + q^{\alpha+l} p^{-\alpha+l} - 2q^{2\alpha} p^l - 2q^{2\alpha} - 2q^{2\alpha+l} p^l - 2q^{2\alpha+l} + q^{3\alpha} p^{\alpha+l} + q^{3\alpha+l} p^{\alpha+l},$$

$$E = -p^l - 1 - q^l p^l - q^l + 2q^\alpha p^{\alpha+l} + 2q^\alpha p^\alpha + 2q^{\alpha+l} p^{\alpha+l} + 2q^{\alpha+l} p^\alpha - q^{2\alpha} p^{2\alpha+l} - q^{2\alpha} p^{2\alpha} - q^{2\alpha+l} p^{2\alpha+l} - q^{2\alpha+l} p^{2\alpha},$$

and it is easily checked that $D = (-q^\alpha p^{-\alpha})E$. As a result,

$$\rho = -q^\alpha p^{-\alpha}. \tag{20}$$

So we conclude that the thus found λ and ρ in (19) and (20) do solve (17), and do not depend on n . Hence, we have proven the proposition.

3. A case for generalizing the Fibonacci property

In this section, we will examine, from the viewpoint of the (non)validity of the FP, the μ -deformed oscillator formulated in [10]. We will prove that the FP is not valid in that case. Then our next goal is to find the generalization of the FP appropriate for the μ -oscillator. For proper extension, we adopt the so-called QF property encapsulated in the QF relation satisfied by the members of the QF sequence. But, first let us recall the necessary setup of the μ -deformed oscillator.

3.1. Nonlinear μ -deformed oscillator

The μ -oscillator [10] is defined in terms of unital algebra whose generating elements a, a^\dagger and N satisfy

$$[N, a] = -a, \quad [N, a^\dagger] = a^\dagger, \quad [N, aa^\dagger] = [N, a^\dagger a] = 0,$$

$$aa^\dagger - a^\dagger a = \varphi_\mu(N + 1) - \varphi_\mu(N),$$

and its structure function is given as

$$a^\dagger a = \varphi_\mu(N) \equiv \frac{N}{1 + \mu N}, \quad aa^\dagger = \varphi_\mu(N + 1) \equiv \frac{N + 1}{1 + \mu(N + 1)} \tag{21}$$

with μ a deformation parameter. That is, the basic commutation relations for the μ -oscillator are equation (3) and

$$[a, a^\dagger] = \frac{N + 1}{1 + \mu(N + 1)} - \frac{N}{1 + \mu N}. \tag{22}$$

In the μ -deformed version of Fock space, with the normalized ground state $|0\rangle$ such that

$$a|0\rangle = 0, \quad N|0\rangle = 0, \quad \varphi_\mu(0) = 0, \tag{23}$$

we have the infinite set of basis states

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{\varphi_\mu(n)!}}|0\rangle, \quad \langle n|m\rangle = \delta_{mn}, \quad n, m = 0, 1, 2, \dots, \tag{24}$$

$\varphi_\mu(n)! = \varphi_\mu(n)\varphi_\mu(n - 1) \dots \varphi_\mu(1)$ on which

$$N|n\rangle = n|n\rangle, \quad \varphi_\mu(N)|n\rangle = \varphi_\mu(n)|n\rangle,$$

$$\langle n - 1|a|n\rangle = \langle n|a^\dagger|n - 1\rangle = \sqrt{\varphi_\mu(n)} = \left(\frac{n}{1 + \mu n}\right)^{\frac{1}{2}}.$$

Since

$$\varphi_\mu(n) = \frac{n}{1 + \mu n}, \tag{25}$$

the energy spectrum of this model reads

$$E_n = \frac{1}{2}(\varphi_\mu(n+1) + \varphi_\mu(n)) = \frac{1}{2} \left(\frac{n+1}{1 + \mu(n+1)} + \frac{n}{1 + \mu n} \right), \tag{26}$$

where the deformation parameter μ is assumed to satisfy $\mu \geq 0$. Setting $\mu = 0$ recovers the known formulae for the standard quantum harmonic oscillator.

3.2. Non-Fibonacci nature of the μ -oscillator

Suppose the μ -oscillator, with $\varphi(N)$ given in (21), obeys the FP stating that

$$E_{n+1} - \lambda E_n - \rho E_{n-1} = 0 \tag{27}$$

with the constants λ and ρ . Taking into account (26), this is rewritten as

$$\begin{aligned} \frac{n+2}{1 + \mu(n+2)} + \frac{n+1}{1 + \mu(n+1)} - \lambda \left(\frac{n+1}{1 + \mu(n+1)} + \frac{n}{1 + \mu n} \right) \\ - \rho \left(\frac{n}{1 + \mu n} + \frac{n-1}{1 + \mu(n-1)} \right) = 0. \end{aligned} \tag{28}$$

Then the following statement is true.

Proposition 2. *The μ -oscillator is not the Fibonacci one, i.e. (28) fails for it.*

To prove this, we take into account the equality, stemming from (28), of the two polynomials one of which is zero. This leads to the following set of equations:

$$\begin{aligned} n^4 : -1 + \lambda + \rho &= 0; \\ n^3 : \lambda(3 + 2\mu) + \rho(3 + 2\mu) - 3 - 2\mu &= 0; \\ n^2 : \lambda(6 + 9\mu - 2\mu^2) + \rho(6 + 7\mu - 2\mu^2) - 6 - 11\mu + 2\mu^2 &= 0; \\ n^1 : \lambda(\mu^2(\mu^{-1} - 1) + (\mu^{-1} + 2)(2\mu^{-2} + \mu^{-1} - 2)) + \rho(-\mu^2(\mu^{-1} + 1) \\ &\quad + (\mu^{-1} + 2)(2\mu^{-2} - \mu^{-1} - 2)) - 2 - 10\mu + \mu^2 + 4\mu^3 = 0; \\ n^0 : 2\mu^2 - 2 + \mu(1 + 2\mu)[(\lambda - 1)(\mu^{-1} - 1) + \rho(\mu^{-1} + 1)] &= 0. \end{aligned}$$

We have to solve this set for λ and ρ . The first two equations yield $\lambda + \rho = 1$. However, inserting this into the rest of the equations leads to inconsistency, as there is no solution satisfying the whole system, for arbitrary μ . So we conclude that the μ -oscillator with $\varphi(n)$ and E_n from (25)–(26) does not possess the FP (27), with the constants λ and ρ .

Therefore, we have to find a modification of the FP which would be adequate for the μ -oscillator and also for some models that extend it.

Possible ways of generalizing the FP

Among the possible modifications, we could try an extended version of the usual Fibonacci relation (1) obtained merely by adding extra terms. For example, we could consider the k -term extended (so-called k -bonacci) relation of the form (see e.g. [4, 5])

$$E_n = \alpha_1 E_{n-1} + \alpha_2 E_{n-2} + \alpha_3 E_{n-3} + \dots + \alpha_k E_{n-k}.$$

For the μ -oscillator, let us check the first extended case of $k = 3$ (so-called Tribonacci relation). It is proved that the μ -oscillator does not satisfy the Tribonacci relation. The proof is in complete analogy with the above case of $k = 2$, see proposition 2: by deriving the

corresponding system of (now six) equations and then trying to solve that system. As a result, we are led to the inconsistency of the system and thus to the conclusion that the μ -oscillator does not satisfy the Tribonacci relation. Moreover, this negative result for the μ -oscillator extends to all other cases, $k \geq 3$, of the k -bonacci relation.

Next, the most radical possibility would be to search for some nonlinear generalization of (1), say in the form $E_{n+1} = F(E_n, E_{n-1})$ with $F(x, y)$ an appropriate function [3]. We prefer, however, to *preserve both the linearity and the three-term form* of relation. But some price should be payed for such a choice, and the price is nothing but the loss of constant nature of λ and ρ . In other words, these coefficients inevitably should be n -dependent.

3.3. Quasi-Fibonacci property of μ -deformed oscillator

We will call ‘QF’ oscillators those deformed oscillators whose energy spectrum satisfies the extended or QF relation involving $\lambda = \lambda(n) \equiv \lambda_n$ and $\rho = \rho(n) \equiv \rho_n$:

$$E_{n+1} = \lambda_n E_n + \rho_n E_{n-1}. \tag{29}$$

We will show that the μ -deformed oscillator belongs to the set of QF oscillator models. That is, by passing from the FP to the QF one, we will prove that it is the QF property which is adequate to the energy spectrum of the μ -oscillator.

There exist three different ways of finding λ_n and ρ_n needed to prove the QF property.

3.3.1. *First way of finding λ_n, ρ_n (by the ‘splitting’ ansatz).* To start with, let us recall that the energy spectrum E_n of the deformed oscillator consists of two terms given by the structure function $\varphi(n)$:

$$E_n = \frac{1}{2}(\varphi(n) + \varphi(n + 1)). \tag{30}$$

From (29) and (30), using ‘splitting’, we have the following system of equations:

$$\begin{cases} \varphi(n + 1) = \lambda_n \varphi(n) + \rho_n \varphi(n - 1), \\ \varphi(n + 2) = \lambda_n \varphi(n + 1) + \rho_n \varphi(n). \end{cases} \tag{31}$$

Solving the latter system for λ_n and ρ_n , we have

$$\lambda_n = \frac{\varphi(n + 1) - \rho_n \varphi(n - 1)}{\varphi(n)}, \quad \rho_n = \frac{\varphi(n + 2)\varphi(n) - \varphi^2(n + 1)}{\varphi^2(n) - \varphi(n + 1)\varphi(n - 1)}. \tag{32}$$

From (32), using expression (25) for $\varphi(n)$ of the μ -oscillator, we finally obtain

$$\lambda_n = 2 \frac{(1 + \mu(1 + 2n))}{(1 + 2\mu n)} \cdot \frac{(1 + \mu n)}{1 + \mu(n + 2)}, \tag{33}$$

$$\rho_n = - \frac{(1 + 2\mu(n + 1))}{(1 + 2\mu n)} \cdot \frac{(1 + \mu n)(1 + \mu(n - 1))}{(1 + \mu(n + 2))(1 + \mu(n + 1))}. \tag{34}$$

It is clear that, for consistency, from (33) and (34) in the limit $\mu \rightarrow 0$, we should recover the *constants*

$$\lim_{\mu \rightarrow 0} \lambda_n = 2, \quad \lim_{\mu \rightarrow 0} \rho_n = -1, \tag{35}$$

i.e. the values λ, ρ of the usual harmonic oscillator. Obviously, that is true. Hence, we conclude that the μ -oscillator satisfies (29) with (26) and (33), (34), and thus is a QF one.

3.3.2. *Second way to find λ_n and ρ_n (by the substitution ansatz).* Recall that the most general form of defining commutation relation appears as [13]

$$\varphi(N + 1) - F(N)\varphi(N) = G(N). \tag{36}$$

By treating this as the recursion relation and fixing the initial conditions

$$\varphi(0) = 0, \quad \varphi(1) = G(0) = \varepsilon, \quad \varepsilon \in \mathfrak{R}, \tag{37}$$

the structure function $\varphi(n)$ can be obtained as

$$\varphi(n) = F(n - 1)! \left(\varepsilon + \sum_{j=1}^{n-1} \frac{G(j)}{F(j)!} \right), \tag{38}$$

where $F(j)! = F(j)F(j - 1) \cdots F(1)$ and $F(0)! = 1$ (at $\varepsilon = 0$, the corresponding formula has been given in [13]).

For our aim, we proceed in analogy with (36)–(38). So we use the ansatz

$$\rho_n = \lambda_{n-1} \quad \text{or} \quad \rho_{n+1} = \lambda_n. \tag{39}$$

With this, the QF relation (29) is rewritten in the form

$$E_{n+2} = \lambda_{n+1} E_{n+1} + \lambda_n E_n \tag{40}$$

or, for a more explicit analogy with (36), in the form

$$\lambda_{n+1} - \left(-\frac{E_n}{E_{n+1}} \right) \lambda_n = \frac{E_{n+2}}{E_{n+1}}, \quad n \geq 1. \tag{41}$$

Fixing the initial conditions as

$$\lambda_0 = \lambda(0) = 0, \quad \lambda_1 = \lambda(1) = \frac{E_2}{E_1}, \tag{42}$$

we derive for λ_n the expression

$$\lambda_n = \left(-\frac{E_{n-1}}{E_n} \right)! \sum_{j=0}^{n-1} \frac{E_{j+2}/E_{j+1}}{(-E_j/E_{j+1})!}. \tag{43}$$

The coefficient ρ_n then follows from (39). In another way, we may use relation (41) to deduce λ_n recursively. Recall that the lowest values of λ_n besides (42) are

$$\begin{aligned} \lambda_2 &= -1 + \frac{E_3}{E_2}, & \lambda_3 &= -1 + \frac{E_4 + E_2}{E_3}, \\ \lambda_4 &= -1 + \frac{E_5 + E_3 - E_2}{E_4}, & \lambda_5 &= -1 + \frac{E_6 + E_4 - E_3 + E_2}{E_5}, \end{aligned}$$

and the formula for the generic λ_n (which is equivalent to (43)) is

$$\lambda_n = -1 + \frac{E_{n+1} + E_{n-1} + \sum_{j=2}^{n-2} (-1)^{n-k+1} E_j}{E_n} = \frac{1}{E_n} \sum_{j=2}^{n+1} (-1)^{n-j+1} E_j. \tag{44}$$

Let us remark that instead of the initial condition $\lambda(0) = 0$ in equation (42), we can set a more general one, namely

$$\lambda_0 = \lambda(0) = c, \quad c = E_1/E_0. \tag{45}$$

Then it follows from (40) that

$$\lambda_1 = \frac{E_2 - cE_0}{E_1}, \quad \lambda_2 = \frac{E_3 - E_2 + cE_0}{E_2}, \quad \lambda_3 = \frac{E_4 - E_3 + E_2 - cE_0}{E_3},$$

and by applying induction, we arrive at the desired result

$$\lambda_n = \frac{\sum_{k=2}^{n+1} (-1)^{n-k+1} E_k + (-1)^n c E_0}{E_n}. \tag{46}$$

Of course, putting $c = 0$ reproduces (44). Using $c = E_1/E_0$ in (45), formula (46) takes the form

$$\lambda_n = (-1)^{n+1} \frac{1}{E_n} \sum_{k=1}^{n+1} (-1)^k E_k \tag{47}$$

which after substitution of (26) yields the desired result:

$$\lambda_n = \left\{ \frac{(-1)^n}{1 + \mu} + \frac{n + 2}{1 + \mu(n + 2)} \right\} \frac{(1 + \mu n)(1 + \mu(n + 1))}{n(1 + \mu(n + 1)) + (n + 1)(1 + \mu n)}, \tag{48}$$

$$\rho_n = \left\{ \frac{(-1)^{n-1}}{1 + \mu} + \frac{n + 1}{1 + \mu(n + 1)} \right\} \frac{(1 + \mu(n - 1))(1 + \mu n)}{(n - 1)(1 + \mu n) + n(1 + \mu(n - 1))}. \tag{49}$$

The solution (48)–(49) of equation (29) obviously differs from the solution (33)–(34) as the ansatz applied in subsection 3.3.2 is essentially different from the one used in 3.3.1.

3.3.3. *Third (most general) approach for deriving λ_n and ρ_n .* We first remark that the formulae derived in the preceding two subsections do not constitute the general result. The goal of this subsection is just to find λ_n and ρ_n from the QF relation (29) within the most general approach. If we take, instead of (29), the system

$$\begin{cases} E_{n+1} = \lambda_n E_n + \rho_n E_{n-1} \\ E_{n+2} = \lambda_{n+1} E_{n+1} + \rho_{n+1} E_n, \end{cases} \tag{50}$$

(as two equations with four unknowns), we cannot calculate λ and ρ as before, see (17)–(18).

To find explicitly λ_n and ρ_n satisfying (29) along with (26), that is, the relation

$$\frac{n + 2}{1 + \mu(n + 2)} = (\lambda_n - 1) \frac{n + 1}{1 + \mu(n + 1)} + (\lambda_n + \rho_n) \frac{n}{1 + \mu n} + \rho_n \frac{n - 1}{1 + \mu(n - 1)}, \tag{51}$$

we now follow a more general scheme: we replace (51) with the pair of equations

$$\frac{2n + 2}{1 + \mu(n + 2)} = (\lambda_n - 1) \frac{n + 1}{1 + \mu(n + 1)} + \frac{K(n; \mu)(n + 1)}{1 + \mu(n + 1)}, \tag{52}$$

$$\frac{-n}{1 + \mu(n + 2)} = (\lambda_n + \rho_n) \frac{n}{1 + \mu n} + \rho_n \frac{n - 1}{1 + \mu(n - 1)} - \frac{K(n; \mu)(n + 1)}{1 + \mu(n + 1)}. \tag{53}$$

Note that the unspecified function $K = K(n; \mu)$ introduced here, due to its arbitrariness, encapsulates the non-uniqueness of splitting (51) into two equations and thus provides the equivalence of (51) with the pair (52)–(53). Indeed, adding (52) and (53) yields (51), or isolating the last term (with $K(n; \mu)$) in (52) and the identical term in (53), we get the equivalence (51) \leftrightarrow (52) and (53).

Now, (52)–(53) are easily solved for λ_n and ρ_n that yields the desired general result

$$\lambda_n = 1 - K(n; \mu) + 2 \frac{1 + \mu(n + 1)}{1 + \mu(n + 2)}, \tag{54}$$

$$\rho_n = \frac{1 + \mu(n - 1)}{1 + 2(n - 1)(1 + \mu n)} \left\{ K(n; \mu) \frac{2(n + 1)(1 + \mu n) - 1}{1 + \mu(n + 1)} - \frac{4n(1 + \mu(n + 1))}{1 + \mu(n + 2)} \right\}. \tag{55}$$

Remark 1. Various choices of the function $K(n; \mu)$ are of interest. Say, by a properly specified choice $K = K^{(1)}$ (respectively $K = K^{(2)}$), we can reproduce, from (54)–(55), expressions (33)–(34) in subsection 3.3.1 (respectively formulae (48)–(49) in subsection 3.3.2). It is easy to verify that the corresponding choices of $K(n; \mu)$ are

$$K^{(1)} = \frac{1 + 3\mu n(1 + 2\mu) + 2\mu(\mu n^2 + 1)}{(1 + 2\mu n)(1 + \mu(n + 2))}, \tag{56}$$

$$K^{(2)} = \frac{\mu}{1 + \mu(n + 2)} + \frac{(-1)^{n+1}(1 + \mu n)(1 + \mu(n + 1))}{(1 + \mu)\{n(1 + \mu(n + 1)) + (n + 1)(1 + \mu n)\}} + \frac{(n + 1)\{3\mu n + (5n + 1)(1 + \mu n)\}}{(1 + \mu(n + 2))\{n(1 + \mu(n + 1)) + (n + 1)(1 + \mu n)\}}. \tag{57}$$

From viewing the QF relation as a ‘locally-Fibonacci’ property, see the end of the introduction, it follows that the (auxiliary) function $K(n; \mu)$, introduced in order to take into account the non-uniqueness of replacing (51) by the pair (52)–(53), plays the role somewhat resembling gauge freedom in gauge theories: the quantities λ_n and ρ_n depend on $K(n; \mu)$ while the energies E_n do not, see (26) and (50)–(51).

Remark 2. As mentioned above, the usual harmonic oscillator is the Fibonacci oscillator for which $\lambda = 2, \rho = -1$. Its structure function and energy spectrum obviously stem from (25)–(26) if $\mu \rightarrow 0$. However, sending $\mu \rightarrow 0$ in formulae (54)–(55) gives

$$\lambda_n = 3 - K(n; 0), \quad \rho_n = \frac{K(n; 0)(2n + 1) - 4n}{2n - 1}, \tag{58}$$

i.e. there is a residual dependence on n if $K(n; 0) \neq 1$. This looks like a kind of surprise: due to n -dependent λ_n, ρ_n , the usual oscillator *can also be treated as a QF oscillator* (satisfying (29)). Only if $K(n; 0) = 1$, we get $\lambda = 2, \rho = -1$, and the usual oscillator becomes a genuine Fibonacci oscillator.

Concerning the last phrase, it is worth to add the following. From (56) and (57) at $\mu \rightarrow 0$, we have $K^{(1)}|_{\mu=0} = 1$, but $K^{(2)}|_{\mu=0} \neq 1$. This explains the fact that λ_n and ρ_n from (33)–(34) do lead at $\mu \rightarrow 0$ to the values $\lambda = 2$ and $\rho = -1$ of the ordinary oscillator, while those in (48)–(49) do not (the reason is the very ansatz $\rho_n = \lambda_{n-1}$ in (39)).

Remark 3. The extension based on replacing the constants λ, ρ with $\lambda_n = \lambda(n)$ and $\rho_n = \rho(n)$ is of principal value for the QF property. However, for the true QF property to be valid, it is in fact enough that only one of the coefficients depends on n , the other being constant. For instance, consider the case $\lambda_n = \lambda(n)$ and $\rho = -1$ for the considered μ -oscillator. Imposing $\rho = -1$ in (55) yields the specified $K(n; \mu)$ -function:

$$K(n; \mu)_{\rho=-1} = \frac{(1 + \mu(n + 1))\{2\mu^2 n^2(3n + 1) + 4\mu n(3n + 2\mu) + \mu(n - 2) + 6n - 1\}}{(1 + \mu(n - 1))(1 + \mu(n + 2))\{2(n + 1)(1 + \mu n) - 1\}}.$$

With this particular K -function, we find λ_n related to $\rho = -1$:

$$\lambda(n)_{\rho=-1} = 1 - \frac{(1 + \mu(n + 1))\{10\mu n^2(2 + \mu n) + (1 - 4\mu)(3\mu n + 1) + 12n\}}{(1 + \mu(n - 1))(1 + \mu(n + 2))\{2(n + 1)(1 + \mu n) - 1\}}. \tag{59}$$

It is worth mentioning that a similar form of the QF relation, with $\rho = -1$ and certain

$\lambda = \lambda(n)$, appears in [30], where the members of the QF sequence therein have the physical sense of (the Fourier transform of) the oscillation amplitude of dipole with number n in the non-periodic chain of dipoles.

Likewise, we may impose $\lambda = 2$, and then find from (54)–(55) the relevant K -function and the corresponding $\rho = \rho_n$:

$$K(n; \mu)_{\lambda=2} = 2 \frac{(1 + \mu(n + 1))}{(1 + \mu(n + 2))} - 1,$$

$$\rho(n)_{\lambda=2} = - \frac{1 + \mu(n - 1)}{1 + 2(n + 1)(1 + \mu n)} \cdot \frac{n(2 + 5\mu) + 4\mu n(n + \mu) + 2\mu^2 n^2(n + 3) - 1}{(1 + \mu(n + 1))(1 + \mu(n + 2))}. \quad (60)$$

Remark 4. Let K in (54), (55) be equal to zero. Then $\lambda_n = \frac{P_1}{Q_1}$, where P_1 and Q_1 are linear in n , while $\rho_n = \frac{P_3}{Q_3}$, where P_3 and Q_3 are cubic in n . At $K = 1$, however, λ_n is again the ‘linear/linear’ expression in n , but ρ_n is the ‘quadratic/quadratic’ one. Let us compare the structures of the coefficients λ_n and ρ_n which were obtained in different ways, see subsections 3.3.1–3.3.3. While λ_n, ρ_n in (33), (34) are of the form $\lambda_n = \frac{P_2^{(1)}}{Q_2^{(1)}}$ and $\rho_n = \frac{\tilde{P}_3^{(1)}}{\tilde{Q}_3^{(1)}}$, the structure seen in (48), (49) looks like $\lambda_n = \frac{P_3^{(2)}}{Q_3^{(2)}}$ and $\rho_n = \frac{\tilde{P}_3^{(2)}}{\tilde{Q}_3^{(2)}}$. Likewise, an appropriate choice of the function $K(n; \mu)$ in the general expressions (54), (55) yields another particular case for λ_n, ρ_n such that they also take the form $\frac{P_3^{(3)}}{Q_3^{(3)}}$. Indeed, the choice

$$K = \frac{(1 + \mu(n + 1))(1 + 2(n - 1)(1 + \mu n))}{(1 + \mu(n + 2))(-1 + 2(n + 1)(1 + \mu n))}$$

again leads to the ‘cubic/cubic’ expressions for both λ_n and ρ_n :

$$\lambda_n = 1 + \frac{1 + \mu(n + 1)}{1 + \mu(n + 2)} \cdot \frac{3 + 2(1 + \mu n)(3n + 1)}{1 - 2(n + 1)(1 + \mu n)},$$

$$\rho_n = - \frac{1 + \mu(n - 1)}{1 + 2(n - 1)(1 + \mu n)} \cdot \frac{1 + 2n(1 + \mu(n + 3))}{1 + \mu(n + 2)}.$$

4. Mixed cases of deformed oscillators

It is possible to construct new models of deformed oscillators which are QF extensions of a particular Fibonacci oscillator of section 2, by combining the latter with the μ -oscillator which plays the role of the basic QF ‘building block’. Clearly, the procedure brings in some additional deformation parameters. In this way, we naturally obtain new models with two deformation parameters μ and q : say, the mixed μ –AC case, and also the mixed μ –BM and μ –TD cases. However, since the models of q -deformed oscillators are contained as particular cases in the p, q -family, see sections 2.1–2.3, it is useful to start with the three-parameter, mixed $(\mu; p, q)$ -family of models.

4.1. The three-parameter ‘mixed’ family of $(\mu; p, q)$ -deformed oscillators

This family of deformed oscillator models arises by combining the p, q -oscillator with the μ -oscillator and is given by the structure function

$$\varphi_{\mu, p, q}(n) = \frac{[n]_{p, q}}{1 + \mu n}, \quad [n]_{p, q} \equiv \frac{p^n - q^n}{p - q}. \quad (61)$$

Of course, the choice (61) of ‘mixed’ deformation is not unique: for the ‘mixed’ structure function, one could also use, say, $\psi_{\mu,p,q}(n) = \frac{[n]_{p,q}}{1+\mu[n]_{p,q}}$ or $\chi_{\mu,p,q}(n) = \left[\frac{n}{1+\mu n}\right]_{p,q}$. Our choice, however, (i) is simpler from the viewpoint of further use in applications like those in [31] and (ii) better correlates with the ideas of [10].

So we expect that this $(\mu; p, q)$ -oscillator, with the energy spectrum

$$E_n = \frac{1}{2}(\varphi_{\mu,p,q}(n+1) + \varphi_{\mu,p,q}(n)),$$

obeys the QF relation.

It is clear that the $(\mu; p, q)$ -deformed family cannot satisfy the pure Fibonacci relation, because of its μ -component as a carrier of the QF property; only if $\mu = 0$, it reduces to the pure (p, q) -oscillator for which the status of the Fibonacci oscillator is recovered.

So consider the $(\mu; p, q)$ -oscillator as the QF one. Its QF property will be certified if the explicit expressions for λ_n and ρ_n are found such that the QF relation (29) holds true:

$$\begin{aligned} & \frac{[2]_{p,q}[n+1]_{p,q} - pq[n]_{p,q}}{1 + \mu(n+2)} + \frac{[n+1]_{p,q}}{1 + \mu(n+1)} \\ &= \lambda_n \left(\frac{[n+1]_{p,q}}{1 + \mu(n+1)} + \frac{[n]_{p,q}}{1 + \mu n} \right) + \rho_n \left(\frac{[n]_{p,q}}{1 + \mu n} + \frac{[n-1]_{p,q}}{1 + \mu(n-1)} \right). \end{aligned} \tag{62}$$

Note that on the lhs the identity

$$[n+2]_{p,q} = [2]_{p,q}[n+1]_{p,q} - pq[n]_{p,q} \tag{63}$$

has been used. Instead of (62), we may *equivalently* exploit the following two relations:

$$\begin{aligned} (\lambda_n - 1) \frac{[n+1]_{p,q}}{1 + \mu(n+1)} &= \frac{[2]_{p,q}[n+1]_{p,q}}{1 + \mu(n+2)} - K \frac{[n+1]_{p,q}}{1 + \mu(n+1)}, \\ (\lambda_n + \rho_n) \frac{[n]_{p,q}}{1 + \mu n} + \rho_n \frac{[n-1]_{p,q}}{1 + \mu(n-1)} &= \frac{-pq[n]_{p,q}}{1 + \mu(n+2)} + K \frac{[n+1]_{p,q}}{1 + \mu(n+1)}. \end{aligned}$$

Let us emphasize that the arbitrary function $K = K(n; \mu, p, q)$ involves both the variable n and the three deformation parameters. As before, it is the unspecified function $K(n; \mu, p, q)$ which guarantees the equivalence with (62). From the pair of equations, using $[2]_{p,q} = p + q$ as implied by (61), the desired solution does follow, namely

$$\lambda_n \equiv \lambda_n(\mu, p, q) = 1 - K + [2]_{p,q} \frac{1 + \mu(n+1)}{1 + \mu(n+2)}, \tag{64}$$

$$\begin{aligned} \rho_n \equiv \rho_n(\mu, p, q) &= \frac{1 + \mu(n-1)}{[n]_{p,q}(1 + \mu(n-1)) + [n-1]_{p,q}(1 + \mu n)} \\ &\times \left\{ K \left([n]_{p,q} + \frac{[n+1]_{p,q}(1 + \mu n)}{1 + \mu(n+1)} \right) \right. \\ &\left. - [n]_{p,q} \left(1 + \frac{[2]_{p,q}(1 + \mu(n+1)) + qp(1 + \mu n)}{1 + \mu(n+2)} \right) \right\}. \end{aligned} \tag{65}$$

This constitutes our general result for the $(\mu; p, q)$ -deformed oscillators. From this, by fixing $K(n; \mu, p, q)$, we can generate, for λ_n and ρ_n , various special expressions.

Let us consider some special cases of (64), (65), with fewer deformation parameters.

- (1) If $p = q = 1$ and $\mu \neq 0$, it is easily checked that one recovers expressions (54), (55) in section 3.3 for $\lambda(n)$ and $\rho(n)$ of the pure μ -oscillator, the latter being the carrier of QF property.

(2) If we put $\mu = 0$ in (64) and (65), we have

$$\lim_{\mu \rightarrow 0} \lambda_n(\mu, p, q) = \lambda(p, q) = \lambda = 1 - K(n; p, q) + [2]_{p,q},$$

$$\lim_{\mu \rightarrow 0} \rho_n(\mu, p, q) = \rho(p, q) = \rho = \frac{K(n; p, q)([n+1] + [n]) - [n](1 + pq + [2]_{p,q})}{[n]_{p,q} + [n-1]_{p,q}}.$$

In the latter expressions, λ and ρ still depend on n . However, at $K = 1$, we recover $\lambda = [2]_{p,q}$ and $\rho = -pq$ of the pure (p, q) -oscillator in section 2. The cancellation of the n -dependence in λ , see $K(n; p, q)$, and in ρ follows by applying identity (63).

(3) Let $p = q^{-1}$ and $q \rightarrow 1$. The corresponding values of the coefficients then read

$$\lambda_n = 1 - K(n; 1, 1) + \frac{2(1 + \mu(n+1))}{1 + \mu(n+2)},$$

$$\rho_n = \frac{1 + \mu(n-1)}{1 + 2(n-1)(1 + \mu n)} \left(K(n; 1, 1) \frac{2(n+1)(1 + \mu n) - 1}{1 + \mu(n+1)} - \frac{4n(1 + \mu(n+1))}{1 + \mu(n+2)} \right)$$

and if, moreover, $K = 1$, we have from the latter

$$\lambda_n|_{K=1} = 2 \frac{1 + \mu(n+1)}{1 + \mu(n+2)},$$

$$\rho_n|_{K=1} = \frac{1 + \mu(n-1)}{1 + 2(n-1)(1 + \mu n)} \left(1 - 2n + \mu \left(\frac{4n}{1 + \mu(n+2)} - \frac{n+1}{1 + \mu(n+1)} \right) \right).$$

(4) Putting $\mu = 0, p = q = 1$ in (64)–(65) for λ_n and ρ_n yields

$$\lambda_n|_{\mu=0} = 3 - K, \quad \rho_n|_{\mu=0} = \frac{K(2n+1) - 4n}{2n-1};$$

see remark 2 for more details.

As mentioned at the beginning of this section, since we have at our disposal the explicit formulae (64), (65) for the three-parameter $(\mu; p, q)$ -family, we immediately obtain the corresponding results for the three distinguished two-parameter cases. Let us illustrate this (recall (8), (10), and $[n]_{TD} \equiv nq^{n-1}$ for the AC, BM and TD cases, respectively):

- let $p \rightarrow 1$. In this case, $\varphi_{\mu,AC}(n) = \frac{[n]_{AC}}{1 + \mu n}$ (mixed μ -AC case);
- put $p = q^{-1}$. In this case, $\varphi_{\mu,BM}(n) = \frac{[n]_{BM}}{1 + \mu n}$ (mixed μ -BM case);
- put $p = q$. In this case, $\varphi_{\mu,TD}(n) = \frac{[n]_{TD}}{1 + \mu n}$ (mixed μ -TD case).

We end with the comment on the case of six parameters (= the combined $(\mu; p, q, \alpha, \beta, l)$ -case)

Remark 5. All the above treatment carried out for the three-parameter $(\mu; p, q)$ -family of deformed oscillators can be extended to the case of six-parameter family which combines the five-parametric case (16) and formula (26) of the μ -oscillator case. Also, one can put $l = \alpha$ in order to have an unambiguous definition of the structure function, consistent with the relation $E_n = \frac{1}{2}(\varphi(n) + \varphi(n+1))$.

5. Conclusions and outlook

In this paper, we dealt with two essentially different classes of deformed oscillators: the Fibonacci class and the (much more rich) QF class. In our first result, we have proven that besides the well-studied two-parameter family of p, q -deformed oscillators known [2] as Fibonacci oscillators, there exists the more general (p, q, α, β, l) -family of deformed oscillators, with three additional parameters, which also belongs to the Fibonacci class. What

is rather unexpected, the coefficients λ and ρ for the five-parametric family of deformed oscillators depend on p, q (as in the p, q -deformed case) and else only on one parameter α , from the remaining three parameters.

On the other hand, according to our second main result reflected in proposition 2, the μ -oscillator [10] does not belong in the Fibonacci class, but to the class of QF oscillators whose basic feature is that the sequence of the values of energy levels obeys the QF relation, being linear and three-term (or two-step) as well, but involving the *non-constant* coefficients λ_n and ρ_n which depend on the deformation parameter μ and also on the number n of the energy level E_n .

The peculiar feature of QF oscillators is that the n -dependent coefficients λ_n and ρ_n are obtained non-uniquely, as it was demonstrated with the three different methods of solving equation (29). While the first two ways lead to partial solutions, the third method is more general due to the (arbitrary) function $K(n; \mu)$ being involved.

We have shown that the μ -oscillator is not the unique one possessing the QF property. Indeed, it can be utilized as a basic ingredient for constructing other families of QF deformed oscillators which have additional deformation parameter(s). An example of the three-parameter family of $(\mu; p, q)$ -deformed oscillators obtainable through combining the p, q -oscillator with the μ -oscillator has been considered. From that, particular two-parameter families (the μ, q -oscillator as the Arik–Coon & the μ -type hybrid, the BM & the μ -type hybrid and the TD & the μ -type hybrid) of deformed oscillators naturally follow. We believe our results provide the basis for constructing in a regular manner numerous new models of multi-parameter deformed nonlinear oscillators belonging to different extensions of the Fibonacci class, in particular, those possessing the QF property. It is also worth noting that interesting *classes of QF oscillators* have been explored in our recent paper [5]. The polynomially deformed non-Fibonacci oscillators treated therein are rather amusing as they can be viewed in three different ways: (1) as QF oscillators; (2) as oscillators obeying inhomogeneous Fibonacci relation and (3) as k -bonacci oscillators. Unlike the classes just mentioned, the μ -oscillator studied in the present paper along with its direct extensions admits only the QF type (way) of description.

For each Fibonacci oscillator, there exist diverse QF extensions, see e.g. the note below (61) and remark 5. This fact, all that is said in the preceding paragraph, and the results of sections 3 and 4, clearly demonstrate that the class of QF models based on the novel conception of QF oscillators is very rich and worthy of detailed study. Although exploration of the physical applications of QF oscillators is at the very beginning, we quote the recent work [31] on the μ -Bose gas model employing μ -oscillators. Let us finally note that in view of the existing deformed fermionic Fibonacci oscillators [32], it is worth studying, along the lines of the present paper, the possible models of fermionic QF oscillators, as yet another class.

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